Spring 2018

Algebra

WVU Mathematics Department

WestVirginiaUniversity
DEPARTMENT OF MATHEMATICS

M.S. Advanced/Ph.D. Entrance exam in Algebra

April, 2018

Part	A			B			C			total score
#	1	2	3	4	5	6	7	8	9	
✓	☐	☐	☐	☐	☐	☐	☐	☐	☐	
pages										
score										

Directions:

** **SOLVE A TOTAL OF SIX QUESTIONS: TWO FROM EACH PART A, B AND C** **

- Mark in the table above (put a check mark in the square below the problem number) which of the problems are to be graded; ***otherwise*, regardless of the problems you have worked on, problems 1, 2, 4, 5, 7 and 8 will be graded**.

- Start each solution on a **new sheet of paper**, write the problem number and page number (of the particular problem). The pages should be numbered **separately for each problem** with the first page of each problem having number 1.

- Write the solution on **one side** of the paper and stay **within the borders**.
  Anything written outside the borders will not be taken into account.

- For each solution submitted, write in the table above how many pages you submit.
  Do not submit scratchworks and solutions that are not to be graded.
  Return your solutions with pages in correct order arranged according to problem numbers and together with this cover page.

Part A. Group Theory

• For a given group G and a subgroup H of G, we denote by $\{e\}$, $|G|$, $\mathsf{Z}(G)$ and $[G : H]$, the trivial subgroup of G, the order of G, the center of G, and the index of H in G, respectively.

(1) Let G be a finite p-group, that is, $|G| = p^n$ for some prime number p and positive integer n. If $\mathsf{Z}(G)$ is cyclic, then prove that G has *exactly one normal subgroup of order p*. (You may assume the fact that $H \cap \mathsf{Z}(G) \neq \{e\}$ for each normal subgroup $\{e\} \neq H$ of G).

(2) Let G be a finite group, where $|G| = p^m r$, p is a prime number, $r > 1$ integer and $\gcd(p, r) = 1$. Assume p^m does not divide $(r-1)!$, that is, the factorial $r-1$. Prove that G is *not simple*.

(3) Let G be a finite group, H a subgroup of G and let p be a prime number. If p *divides* $|H|$, and P is the *unique* Sylow p-subgroup of G, then prove that $P \cap H \neq \{e\}$, and $P \cap H$ is the *unique* Sylow p-subgroup of H. (Hint: prove that p does not divide $[H : P \cap H]$).

Part B. Field and Galois Theory

• A Galois extension is a field extension that is finite, normal and separable. If K/F is a Galois extension, then $\mathrm{Gal}(K/F)$ denotes the Galois group of K/F. In the following $\mathbb{C}$ and $\mathbb{Q}$ denote the complex and rational numbers, respectively.

(4) Let E/F be a field extension. Assume $[E : F] = 2$, that is, the degree of E over F is 2.
Prove that E/F is a *normal* extension. Justify your argument clearly.
(Hint: show E is the splitting field of a polynomial over F).

(5) Assume there exists a field F with the following properties:
  (i) $\sqrt{2} \notin F \subseteq \mathbb{C}$.
  (ii) If L is a field such that $F \subseteq L \subseteq \mathbb{C}$ and $F \neq L$, then $\sqrt{2} \in L$.
If K is a field, and $F \subseteq K$ and $K \subseteq \mathbb{C}$ are field extensions such that K/F is Galois, then prove that the group $\mathrm{Gal}(K/F)$ is *cyclic*.

(6) Determine all *intermediate* field(s) of the Galois extension $\mathbb{Q}(\omega)/\mathbb{Q}$, where ω is a *primitive fifth root of unity*. Give a generator of the intermediate field(s) you find.
(Hint: use the fundamental theorem of Galois theory).

Part C. Ring and Module Theory

- Assume all rings are *commutative* and all rings have *multiplicative identity* 1 such that $1 \neq 0$.

(7) Let R be a commutative ring. Assume the following condition holds:
" If I is an ideal of R such that $I \neq R$, then I is a prime ideal of R ".
Prove that R is a *field*.

(8) Let R be a commutative ring. Assume every ideal of R is a *free* R-module.
Prove that R is a *principal ideal domain*.

(9) Let R be a commutative ring. Consider the following *commutative* diagram of R-modules and R-module homomorphisms with exact rows. In other words, we have:

(a) A, B, C, A', B', C' are R-modules, and $\psi, \Phi, \alpha, \beta, \gamma, \psi', \Phi'$ are R-module homomorphisms.

(b) $\beta\psi = \psi'\alpha$ and $\gamma\Phi = \Phi'\beta$.

(c) $\mathrm{im}(\psi) = \ker(\Phi)$ and $\mathrm{im}(\psi') = \ker(\Phi')$.

Here the operation between homomorphisms is the composition, that is, $\beta\psi$ means the composition of β and ψ. Also, im denotes the *image*, and ker denotes the *kernel* of the homomorphism.

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\Phi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "\psi'"] & B' \arrow[r, "\Phi'"] & C'
\end{tikzcd}$$

Assume Φ, α and γ are surjective. Prove that β is surjective.

page # of problem # name: